THE POROUS MEDIUM EQUATION WITH NONLINEAR ABSORPTION AND MOVING BOUNDARIES

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ABSTRACT
We consider a nonlinear parabolic equation containing the porous medium operator and a nonlinear absorption term, which causes the appearance of a moving boundary. Basic behavior and regularity results are obtained for the solution and the moving boundary under two different boundary conditions. Also, the behavior of the solution and the moving boundary as time goes to infinity is investigated.

§1. Introduction

Suppose that $\phi:[0,\infty)\rightarrow[0,\infty)$ is smooth, strictly increasing with $\phi(0)=0$, and consider the pair of nonlinear parabolic moving boundary problems

$$(D)\qquad \begin{aligned} &u_t^D=[\phi(u^D)]_{xx}+f(u^D), \quad t>0, \quad 0<x<\gamma^D(t),\\ &u^D=c_0, \quad x=0 \quad \text{and } u^D=\phi(u^D)_x=0, \quad x=\gamma^D(t), \quad t>0,\\ &u^D=u_0, \quad t=0, \quad 0<x<\sigma, \end{aligned}$$

and

$$(N)\qquad \begin{aligned} &u_t^N=[\phi(u^N)]_{xx}+f(u^N), \quad t>0, \quad 0<x<\gamma^N(t),\\ &\phi(u^N)_x=0, \quad x=0 \quad \text{and} \quad u^N=\phi(u^N)_x=0, \quad x=\gamma^N(t), \quad t>0,\\ &u^N=u_0, \quad t=0, \quad 0<x<\sigma. \end{aligned}$$

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Here it is assumed that $c_0 > 0$, $f:[0, c_0] \to \mathbf{R}$ is C^1 and nonincreasing with $f(0) < 0$, and that u_0 is a given initial value on the x-interval $[0, \sigma]$. It is required that u^D and u^N remain nonnegative for all $t \geqq 0$ and $0 < x \leqq \sigma$, and so the assumption that $f(0) < 0$ generally causes the appearance of the moving boundaries $\gamma^D(t)$ and $\gamma^N(t)$. If u^j, $j = D$ or N, is the concentration of a material in a medium, then $\phi(u^j)_{xx}$ is the diffusion term and $f(u^j)$ the absorption term. When $j = D$ the concentration is held constant at the left boundary and when $j = N$ the left boundary is sealed so that there is no flow out or into the medium at the left boundary. The moving boundary $\gamma^j(t)$ corresponds to the penetration depth of the material.

The case when $\phi(u) = du$ where $d > 0$ has been studied in Crank and Gupta [10] and Lyons and Martin [19]. The main problem studied here is when ϕ is nonlinear [an important case is when $\phi(u) = du^m$ where $d > 0$ and $m > 1$]. Although the general types of behavior for the solution in this case is similar to the semilinear case, there are several difficult technical problems that arise. The principal technique is to apply results from the general theory of nonlinear semigroups to obtain information on the existence and behavior of solutions to this concrete problem.

This paper is divided into three sections. The first part deals with the existence of (generalized) solutions to (D) and (N) and is based on the theory of nonlinear semigroups in the space $\mathscr{L}^1(0, \sigma)$. In particular, solutions are constructed using approximations and the Crandall–Liggett theorem for the generation of semigroups in general Banach spaces (see, e.g., Brezis and Pazy [7], Evans [11], and Goldstein [14]). (See also Fasano and Primicerio [13].) The second part gives information on the behavior of solutions as $t \to \infty$ and, in particular, the asymptotic stability of the equilibrium solutions (see, e.g., Alikakos and Rostamian [1], Aronson, Crandall and Peletier [4], Berryman and Holland [6], Kamenomostskaya [16] and Lyons and Martin [19]). The final part considers certain differentiability properties of the generalized solutions (see, e.g., Aronson [2], Aronson and Benilan [3], Crandall and Pierre [9], Evans [12], Kalashnikov [15], Kruzhkov [17] and Oleinik et al. [20]).

§2. Basic notations and results

The fundamental existence and behavior results for solutions are developed in this section. Throughout this section it is assumed that ϕ is a function with the following properties:

$$\phi : \mathbf{R} \to \mathbf{R} \text{ is } C^2, \quad \phi(0) = 0 \quad \text{and} \quad \phi'(r) > 0 \quad \text{if } r \neq 0. \tag{2.1}$$

Principal examples are the functions $\phi(r) \equiv dr^m$ where $d > 0$ and $m \geqq 1$. In addition it is assumed in this section that δ, σ and c_0 are numbers with $c_0 > \delta \geqq 0$ and $\sigma > 0$. Define the class of functions $\mathscr{F}_\delta$ by

$$\mathscr{F}_\delta \equiv \{f : \mathbf{R} \rightarrow \mathbf{R} : f \text{ is } C^1 \text{ and nonincreasing, } f(\xi) > 0 \text{ for } \xi < \delta \text{ and } f(\xi) < 0 \text{ for } \xi > c_0\} \tag{2.2}$$

and the class of functions $\mathscr{D}_\delta$ by

$$\mathscr{D}_\delta \equiv \{u : [0, \sigma] \rightarrow [\delta, c_0] : u \text{ is measurable and nonincreasing}\}. \tag{2.3}$$

The class $\mathscr{D}_\delta$ is considered as a subset of $\mathscr{L}^1 \equiv \mathscr{L}^1([0, \sigma]; \mathbf{R})$ where

$$|u|_1 \equiv \int_0^\sigma |u(x)| dx \qquad \text{for all } u \in \mathscr{L}^1.$$

Note in particular that $\mathscr{D}_\delta$ is a closed, bounded and convex subset of $\mathscr{L}^1$. The class of functions $\mathscr{F}_\delta$ will be expanded to a wider class later [see (2.17)].

For the first part of this section we consider the nonlinear initial-boundary value problems

$$\begin{cases} u_t = \phi(u)_{xx} + f(u), \quad t > 0, \quad 0 < x < \sigma, \\ u(x, 0) = u_0(x), \quad 0 < x < \sigma, \\ \text{and either} \\ u(0, t) = c_0, \quad u(\sigma, t) = \delta \quad \text{or} \\ \phi(u)_x(0, t) = 0, \quad u(\sigma, t) = \delta, \end{cases} \tag{2.4}$$

where $u_0 \in \mathscr{D}_\delta$ and $f \in \mathscr{F}_\delta$. As opposed to studying (2.4) directly, we set up a corresponding abstract Cauchy problem in the space $\mathscr{L}^1$. First define the porous medium operator P on $\mathscr{L}^1$ by

$$Pu = \phi(u)'' \quad \text{for all } u \in D(P) \equiv \{u \in \mathscr{L}^1 : \phi(u), \phi(u)' \text{ are abs. cont. and } \phi(u)'' \in \mathscr{L}^1\}. \tag{2.5}$$

Now for each δ and $f \in \mathscr{F}_\delta$ define the operators $A^D_{\delta,f}$ and $A^N_{\delta,f}$ by

$$\begin{aligned} D(A^D_{\delta,f}) &= \{u \in \mathscr{D}_\delta \cap D(P) : u(0) = c_0, u(\sigma) = \delta\}, \\ A^D_{\delta,f} u &= \phi(u)'' + f(u) \qquad \text{for all } u \in D(A^D_{\delta,f}), \end{aligned} \tag{2.6}$$

and

$$D(A^N_{\delta,f}) = \{u \in \mathcal{D}_\delta \cap D(P) : \phi(u)'(0) = 0,\ u(\sigma) = \delta\},$$
$$A^N_{\delta,f}u = \phi(u)'' + f(u) \qquad \text{for all } u \in D(A^N_{\delta,f}). \tag{2.7}$$

Unless needed for clarity, the subscripts δ and f are normally omitted in writing the operators in (2.6) and (2.7). Also, it is assumed throughout that $j \in \{D, N\}$ so that A^j denotes the operator in (2.6) if $j = D$ and the operator in (2.7) if $j = N$. Note that the initial-boundary value problems (2.4) can be written as an abstract equation in $\mathcal{L}^1$ in the following manner:

$$u'(t) = A^j_{\delta,f}u(t), \quad t \geqq 0, \quad u(0) = u_0, \quad j \in \{D, N\}. \tag{2.8}$$

In order to apply the Crandall–Liggett theorem on the generation of nonlinear contraction semigroups, we establish the following properties for the resolvents of the operators A^j.

LEMMA 1. *Suppose that $A^j = A^j_{\delta,f}$ is defined by* (2.6) *or* (2.7) *and that $h > 0$. Then $I - hA^j$ is* 1–1 *on $D(A^j)$,*

$$R(I - hA^j) \equiv \{u - hA^ju : u \in D(A^j)\} \supset \mathcal{D}_\delta \tag{2.9}$$

and

$$|(I - hA^j)^{-1}u - (I - hA^j)^{-1}v|_1 \leqq |u - v|_1 \qquad \textit{for all } u, v \in \mathcal{D}_\delta. \tag{2.10}$$

INDICATION OF PROOF. From well-known facts about the dissipativeness of the porous medium operator P in the space $\mathcal{L}^1$ (see, e.g., [11]) together with the assumption that f is nonincreasing, it is easy to show that

$$|(I - hA^j)u - (I - hA^j)v|_1 \geqq |u - v|_1 \tag{2.11}$$

for all $u, v \in D(A^j)$. This shows that $I - hA^j$ is 1–1 and that (2.10) is valid for all $u, v \in \mathcal{R}(I - hA^j)$. Thus, this lemma will be established once it is shown that (2.9) holds. Since A^j is closed [i.e., the set (u, A^ju) with $u \in D(A^j)$ is closed in $\mathcal{L}^1 \times \mathcal{L}^1$] we see from (2.11) that $\mathcal{R}(I - hA^j)$ is closed in $\mathcal{L}^1$, and so it is only necessary to show that $\mathcal{R}(I - hA^j)$ contains a dense subset of $\mathcal{D}_\delta$. Therefore, we show that if z is a continuous member of $\mathcal{D}_\delta$, the equation $(I - hA^j)u = z$ has a solution $u \in D(A^j)$. Assume for definiteness that $j = D$ and consider the equation

$$u(x) - h\left[\frac{d^2}{dx^2}\phi(u(x)) + f(u(x))\right] = z(x), \quad 0 < x < \sigma, \quad u(0) = c_0, \quad u(\sigma) = \delta. \tag{2.12}$$

Defining $\bar{\beta}(\xi)=\phi^{-1}(\xi)$ for $\phi(\delta)-1\leqq\xi\leqq\phi(c_0)+1$, $\bar{\beta}(\xi)=\phi^{-1}(\phi(\delta)-1)$ for $\xi<\phi(\delta)-1$ and $\bar{\beta}(\xi)=\phi^{-1}(\phi(c_0)+1)$ for $\xi\leqq\phi(c_0)+1$, we have that $\bar{\beta}$ is continuous and uniformly bounded. It now follows that the equation

$$\begin{gathered}\bar{\beta}(v)-h[v''+f(\bar{\beta}(v))]=z,\\ v(0)=\phi(c_0),\qquad v(\sigma)=\phi(\delta)\end{gathered}\tag{2.13}$$

has a solution. Observe that if L is the inhomogeneous operator $Lv=-hv''$ with $D(L)=\{v\in\mathscr{L}^1: v,v'$ are abs. cont., $v''\in\mathscr{L}^1$ and $v(0)=\phi(c_0)$, $v(\sigma)=\phi(\delta)\}$, then L has a compact inverse defined on $\mathscr{L}^1$. Therefore, the operator mapping $v\to L^{-1}(\bar{\beta}(v)-hf(\bar{\beta}(v)))$ is completely continuous and uniformly bounded on $\mathscr{L}^1$, and so it has a fixed point by Schauder's Theorem. Any such fixed point is a solution to (2.13). An easy application of the maximum principle shows that any solution v to (2.13) must satisfy $\phi(\delta)\leqq v(x)\leqq\phi(c_0)$ for all $x\in[0,\sigma]$. Thus, $\bar{\beta}(v(x))=\phi^{-1}(v(x))$ for all $x\in[0,\sigma]$ and it follows that $u(x)\equiv\phi^{-1}(v(x))$ is a solution to (2.12) and that $\delta\leqq u(x)\leqq c_0$ for all $x\in[0,\sigma]$. To complete the proof we need to show that u is nonincreasing on $[0,\sigma]$, and since ϕ^{-1} is increasing it suffices to show that the solution v to (2.13) is nonincreasing. Suppose, for contradiction, that there is an $x_0\in(0,\sigma)$ such that $v'(x_0)>0$. Since $v(\sigma)=\phi(\delta)\leqq v(x)\leqq\phi(c_0)=v(0)$ for all $x\in[0,\sigma]$ we can deduce that if

$$x_1=\inf\{x\in[0,x_0]: v'(y)>0 \text{ for } x\leqq y\leqq x_0\}\qquad\text{and}$$

$$x_2=\sup\{x\in[x_0,\sigma]: v'(y)>0 \text{ for } x_0\leqq y\leqq x\}$$

then $v''(x_2)\leqq 0$ and $v''(x_1)\geqq 0$. However, since z is increasing and $\xi\to\phi^{-1}(\xi)-hf(\phi^{-1}(\xi))$ is strictly increasing on $[\phi(\delta),\phi(c_0)]$, we have that

$$v''(x)=-\frac{1}{h}z(x)+\frac{1}{h}[\phi^{-1}(v(x))-hf(\phi^{-1}(v(x)))]$$

is strictly increasing on $[x_1,x_2]$. This is impossible since $v''(x_1)\geqq 0$ and $v''(x_2)\leqq 0$, and we conclude that v (and hence u) is nonincreasing and completes the proof indication of Lemma 1.

Inequalities for solutions also play an important role in these techniques. If $u,v\in\mathscr{L}^1$ we write $u\leqq v$ only in case $u(x)\leqq v(x)$ for almost all $x\in[0,\sigma]$. If $f_1,f_2\in\mathscr{F}_\delta$ then $f_1\leqq f_2$ only in case $f_1(\xi)\leqq f_2(\xi)$ for all ξ. Note that if $\delta_1\leqq\delta_2$ then $\mathscr{F}_{\delta_1}\supset\mathscr{F}_{\delta_2}$, and so it makes sense to write $f_1\leqq f_2$ whenever $f_1\in\mathscr{F}_{\delta_1}$, $f_2\in\mathscr{F}_{\delta_2}$ and $\delta_1\leqq\delta_2$.

LEMMA 2. *Suppose that* $j \in \{D, N\}$, *that* $(\delta_i, f_i) \in [0, c_0) \times \mathcal{F}_{\delta_i}$ *and* $w_i \in \mathcal{D}_{\delta_i}$ *for* $i = 1,2$. *Suppose further that* $\delta_1 \leqq \delta_2$, $f_1 \leqq f_2$ *and* $w_1 \leqq w_2$. *Then*

$$(I - hA^j_{\delta_1, f_1})^{-1} w_1 \leqq (I - hA^j_{\delta_2, f_2})^{-1} w_2 \qquad \text{for all } h > 0. \tag{2.14}$$

REMARK 2.1. Note that Lemma 2 essentially asserts that the map $(\delta, f, w) \rightarrow (I - hA^j_{\delta, f})^{-1} w$ of $[0, c_0) \times \mathcal{F}_\delta \times \mathcal{D}_\delta$ into $\mathcal{L}^1$ is increasing in each of its components for any $h > 0$.

PROOF OF LEMMA 2. Set $\beta = \phi^{-1}$, $B_i = A^j_{\delta_i, f_i}$ and $u_i = (I - hB_i)^{-1} w_i$ for $i = 1,2$. Setting $v_i = \phi(u_i)$, we have $\beta(v_i) - h[v_i'' + f_i(\beta(v_i))] = w_i$ for $i = 1,2$. Assume first that w_1 and w_2 are continuous and let $x_0 \in [0, \sigma]$ be such that

$$v_1(x_0) - v_2(x_0) = \max_{0 \leqq x \leqq \sigma} \{v_1(x) - v_2(x)\}.$$

Suppose, for contradiction, that $v_1(x_0) - v_2(x_0) > 0$. The continuity of w_1 and w_2 and the boundary conditions are seen to imply that $v_1''(x_0) - v_2''(x_0) \leqq 0$, and since $w_1(x_0) \leqq w_2(x_0)$ it follows that

$$\begin{aligned} 0 &\leqq -h(v_1''(x_0) - v_2''(x_0)) \\ &= w_1(x_0) - w_2(x_0) - [(I - hf_1)(\beta(v_1(x_0))) - (I - hf_2)(\beta(v_2(x_0)))] \\ &\leqq -(I - hf_1)(\beta(v_1(x_0))) + (I - hf_2)(\beta(v_2(x_0))). \end{aligned}$$

However, $f_2 \geqq f_1$ implies $I - hf_2 \leqq I - hf_1$, and so

$$(I - hf_1)(\beta(v_1(x_0))) \leqq (I - hf_2)(\beta(v_2(x_0))) \leqq (I - hf_1)(\beta(v_2(x_0))).$$

This is impossible since $I - hf_1$ is strictly increasing and $v_1(x_0) > v_2(x_0)$ implies that $\beta(v_1(x_0)) > \beta(v_2(x_0))$. Therefore, $v_1 \leqq v_2$ and hence

$$u_1 = \beta(v_1) \leqq \beta(v_2) = u_2.$$

Since every w_1 and w_2 can be approximated by continuous $\bar{w}_1 \in \mathcal{D}_{\delta_1}$ and $\bar{w}_2 \in \mathcal{D}_{\delta_2}$ with $\bar{w}_1 \leqq w_1 \leqq w_2 \leqq \bar{w}_2$, we see that Lemma 2 is true.

The results of Lemma 1 show that A^j is a dissipative operator on $D(A^j)$ and $R(I - hA^j) \supset D_\delta = D(A^j)$ for each $h > 0$. By the Crandall–Liggett theorem [8],

$$S^j_{\delta, f}(t) u \equiv \lim_{n \to \infty} \left(I - \frac{t}{n} A^j_{\delta, f} \right)^{-n} u, \qquad t \geqq 0, \quad u \in \mathcal{D}_\delta \tag{2.15}$$

exists, and $S^j = S^j_{\delta, f}$ is a (C_0) contraction semigroup of nonlinear operators on $\mathcal{D}_\delta$:

(S1) $S^j(0)u = u$, $S^j(t+s)u = S^j(t)S^j(s)u$ for all $t, \delta \leqq 0$, $u \in \mathscr{D}_\delta$;
(S2) $t \to S^j(t)u$ is continuous from $[0,\infty)$ into $\mathscr{L}^1$ for each $u \in \mathscr{D}_\delta$;
(S3) $|S^j(t)u - S^j(t)v|_1 \leqq |u - v|_1$ for all $t \geqq 0$, $u, v \in \mathscr{D}_\delta$,
(S4) $S^j(t)u \leqq S^j(t)v$ for all $t \geqq 0$ whenever $u, v \in \mathscr{D}_\delta$ with $u \leqq v$.

The order preserving property (S4) follows easily from Lemma 2 and the exponential formula (2.15). In fact, it is easy to see that the following is valid.

LEMMA 3. *Suppose that $j \in \{D, N\}$, that $0 \leqq \delta_1 \leqq \delta_2 < c_0$, that $f_1 \in \mathscr{F}_{\delta_1}$ and $f_2 \in \mathscr{F}_{\delta_2}$ with $f_1 \leqq f_2$, and that $u_1 \in \mathscr{D}_{\delta_1}$ and $u_2 \in \mathscr{D}_{\delta_2}$ with $u_1 \leqq u_2$, then*

$$S^j_{\delta_1, f_1}(t)u_1 \leqq S^j_{\delta_2, f_2}(t)u_2 \qquad \textit{for all } t \geqq 0. \tag{2.16}$$

For each $u_0 \in \mathscr{D}_\delta$, the function u defined on $[0,\sigma] \times [0,\infty)$ by $u(x,t) = [S^j_{\delta,f}(t)u_0](x)$ is a generalized solution to (2.4) satisfying the boundary conditions $u(0,t) = c_0$, $u(\sigma,t) = \delta$ if $j = D$ and the boundary conditions $\phi(u)_x(0,t) = 0$, $u(\sigma,t) = \delta$ if $j = N$. A crucial point in these results is that $f(\delta) \geqq 0$, and hence the solutions naturally remain larger than δ by the maximum principle. In fact, when $\delta > 0$ and the initial value u_0 is sufficiently smooth, the generalized solution defined by the semigroup $S^j_{\delta,f}$ is actually a classical solution to (2.4).

LEMMA 4. *Suppose that $\delta > 0$, $f \in \mathscr{F}_\delta$ and $u_0 \in D(A^j_{\delta,f})$ is infinitely differentiable. If*

$$u(x,t) \equiv [S^j_{\delta,f}(t)](x) \qquad \textit{for } (x,t) \in [0,\sigma] \times [0,\infty),$$

then u is the classical solution to (2.4). *Also, $u(x,t)$ is C^2 in x and C^1 in t on $[0,\sigma] \times [0,\infty)$, and $(x,t) \to \partial_{xxx}u(x,t)$ exists and is continuous on $(0,\sigma) \times (0,\infty)$.*

PROOF. Let $\phi_\delta \in C^2(\mathbf{R})$ be such that $\phi'_\delta(u) > 0$ for all $u \in \mathbf{R}$ and $\phi_\delta(u) = \phi(u)$ for $|u| \geqq \delta$. By Theorem 5.2, p. 564, and the arguments from p. 516 of Ladyzenskaja et al. [17], there exists a unique classical solution $u(x,t)$ to

$$u_t = \phi_\delta(u)_{xx} + f(u),$$

$$u(0,t) = c_0, \qquad u(t,\sigma) = \delta,$$

$$u(x,0) = u_0(x),$$

and that $\partial_{xxx}u(x,t)$ is continuous on $(0,\sigma) \times (0,\infty)$. Since $f(c_0) \leqq 0$ and $f(\delta) \geqq 0$, the maximum principle (see Protter and Weinberger [21, Theorem 3.12]) implies that $\delta \leqq u(x,t) \leqq c_0$ for all $(x,t) \in [0,\sigma] \times [0,\infty)$. Thus, $\phi_\delta(u(x,t)) \equiv \phi(u(x,t))$ and u is a classical solution to (2.4). Noting that the function $u : [0,\infty) \to \mathscr{L}^1$ defined by $[u(t)](x) = u(x,t)$ is an $\mathscr{L}^1$-solution to (2.8) with $j = D$, and then applying Theorem 1.5, p. 115, of Barbu [5], shows that $u(t) = S^j_{\delta,f}(t)u_0$ and since the case $j = N$ can be handled similarly, this completes the proof of Lemma 4.

Our main concern is to consider equations of type (2.4) with a large class of nonlinear terms f. In particular, we remove the restriction that $f(0) \geqq 0$ when $\delta = 0$, so that classical solutions to (2.4) do not necessarily remain positive. Define the class of functions by

$$\mathcal{F} = \{f : \mathbf{R} \to \mathbf{R} : f \text{ in } C^1, \text{ nonincreasing, and } f(\xi) < 0 \text{ for all } \xi > c_0\}. \tag{2.17}$$

It is clear that $\mathcal{F} \supset \mathcal{F}_\delta$ for each $\delta \geqq 0$. Also, if $f \in \mathcal{F}$ with $f(\delta) \geqq 0$ then f, modified so that $f(\xi) > 0$ for $\xi < \delta$, is in $\mathcal{F}_\delta$. Allowing $f(0) < 0$ leads to the following moving boundary problem:

$$\begin{aligned} &u^j_t(x,t) = \phi(u^j(x,t))_{xx} + f(u^j(x,t)), \quad t > 0, \quad 0 < x < \gamma^j(t) \leqq \sigma, \\ &u^j(x,0) = u_0(x), \quad 0 < x < \sigma, \\ &u^j(\gamma^j(t),t) = \phi(u^j(\gamma^j(t),t))_x = 0 \quad \text{if } t > 0, \quad \gamma^j(t) < \sigma, \\ &u^j(x,t) = 0 \quad \text{if } \gamma^j(t) \leqq x \leqq \sigma, \\ &u^j(0,t) = c_0 \quad \text{if } j = D \quad \text{and} \quad \phi(u^j(0,t))_x = 0 \quad \text{if } j = N, \quad t > 0. \end{aligned} \tag{2.18}$$

In order to write (2.18) as an abstract Cauchy problem it is necessary to consider *multivalued* operators. Define the domains $\mathcal{D}^D$ and $\mathcal{D}^N$ by

$$\begin{aligned} &\mathcal{D}^D = \{u \in \mathcal{D}_0 \cap D(P) : u(0) = c_0, u(\sigma) = 0\} \qquad \text{and} \\ &\mathcal{D}^N = \{u \in \mathcal{D}_0 \cap D(P) : \phi(u)'(0) = 0, u(\sigma) = 0\}. \end{aligned} \tag{2.19}$$

Notice that $\mathcal{D}^j = D(A^j_{0,f})$ for $j \in \{D, N\}$ and any $f \in \mathcal{F}_0$. Now define the multivalued operators A^j_f for $j \in \{D, N\}$ and $f \in \mathcal{F}$ as follows:

$$\begin{aligned} &(u,v) \in A^j \text{ only in case } u \in \mathcal{D}^j, \\ &v(x) = \phi(u)''(x) + f(u(x)) \text{ for almost all } x \in [0,\sigma] \\ &\text{such that } u(x) > 0, \ v(x) = 0 \text{ if } u(x) = 0 \text{ and } f(0) \geqq 0, \\ &\text{and } f(0) \leqq v(x) \leqq 0 \text{ if } u(x) = 0 \text{ and } f(0) < 0. \end{aligned} \tag{2.20}$$

If $f \in \mathcal{F}_0$ then $A^j_f = A^j_{0,f}$ and we see that A^j_f is multivalued only in case $f(0) < 0$. Note also that if $f \in \mathcal{F}$ and we define the relation $\bar{f}$ by

(2.21)
$$\bar{f}(\xi)=\begin{cases}\{f(\xi)\} & \text{if } \xi>0,\\ \{f(0)\} & \text{if } \xi=0 \text{ and } f(0)\geqq 0,\\ [f(0),0] & \text{if } \xi=0 \text{ and } f(0)<0,\end{cases}$$

then

(2.22)
$$(u,v)\in A_f^j \text{ only in case } u\in\mathscr{D}^j \quad \text{and}$$
$$v(x)\in\phi(u)''(x)+\bar{f}(u(x)) \text{ for almost all } x\in[0,\sigma].$$

We have the following fundamental result for these operators A_f^j:

THEOREM 1. *Suppose that* $j\in\{D,N\}$, $h>0$ *and* $f\in\mathscr{F}$. *Then* $I-hA_f^j$ *is* 1–1, $\mathscr{R}(I-hA_f^j)\supset\mathscr{D}_0=\mathscr{D}^j$,

(2.23) $$|(I-hA_f^j)^{-1}u-(I-hA_f^j)^{-1}v|_1\leqq|u-v|_1 \quad \textit{for all } u,v\in\mathscr{D}_0, \quad \textit{and}$$

(2.24) $$(I-hA_f^j)^{-1}u\geqq(I-hA_f^j)^{-1}v \quad \textit{for all } u,v\in\mathscr{D}_0 \textit{ with } u\geqq v.$$

Furthermore,

(2.25) $$S_f^j(t)u\equiv\lim_{n\to\infty}\left(I-\frac{t}{n}A_f^j\right)^{-n}u, \qquad t\geqq 0, \quad u\in\mathscr{D}_0$$

exists in $\mathscr{L}^1$ *and* S_f^j *has the following properties*:

(S1) $S_f^j(0)u=u$, $S_f^j(t+s)u=S_f^j(t)S_f^j(s)u$ *for all* $u\in\mathscr{D}_0$ *and* $t,s\geqq 0$,

(S2) $t\to S_f^j(t)u$ *is continuous from* $[0,\infty)$ *into* $\mathscr{L}^1$ *for each* $u\in\mathscr{D}_0$,

(S3) $|S_f^j(t)u-S_f^j(t)v|_1\leqq|u-v|_1$ *for all* $t\geqq 0$, $u,v\in\mathscr{D}_0$,

(S4) $S_f^j(t)u\leqq S_f^j(t)v$ *for all* $t\geqq 0$ *whenever* $u,v\in\mathscr{D}_0$ *with* $u\leqq v$.

Observe that the properties (S1)–(S4) for the semigroup S_f^j are precisely the same as those for the semigroup $S_{0,f}^j$ given previously. Since the porous medium operator P is dissipative and the multivalued extension $\bar{f}$ of f in (2.21) is also dissipative, it is easy to check that A_f^j is dissipative and hence

$$|(I-hA_f^j)u-(I-hA_f^j)v|_1\geqq|u-v|_1 \qquad \text{for all } u,v\in\mathscr{D}^j, \quad h>0.$$

This shows that (2.23) holds whenever u and v are in $\mathscr{R}(I-hA_f^j)$. Thus the main part of the proof of Theorem 1 is to show that $\mathscr{R}(I-hA_f^j)\supset\mathscr{D}_0$, and to accomplish this we use the following lemma:

LEMMA 5. *Suppose that* $f\in\mathscr{F}$ *and* $v\in\mathscr{D}_0$. *Suppose further that* $\{\delta_k\}_1^\infty$ *is a nonincreasing sequence in* $[0,c_0)$ *with* $\lim_{k\to\infty}\delta_k=0$ *and that* $\{f_k\}_1^\infty$ *is a sequence of functions with* $f_k\in\mathscr{F}_{\delta_k}$, $f_{k+1}\leqq f_k$ *for* $k\geqq 1$, *and*

(2.26) $\lim_{k\to\infty} f_k(\xi) = f(\xi)$ *uniformly for* $\xi \in [\eta, c_0]$ *for each* $\eta > 0$.

Also let $\{v_k\}_1^\infty$ *be any sequence in* $\mathcal{L}^1$ *such that* $v_k \in \mathcal{D}_{\delta_k}$, $v_{k+1} \leqq v_k$ *for* $k \geqq 1$, *and*

$$\lim_{k\to\infty} | v_k - v |_1 = 0.$$

Then it follows that $v \in \mathcal{R}(I - hA_f^j)$, *that*

$$(2.27)\qquad (I - hA^j_{\delta_{k+1}, f_{k+1}})^{-1} v_{k+1} \leqq (I - hA^j_{\delta_k, f_k})^{-1} v_k \qquad \text{for } h > 0, \quad k \geqq 1,$$

and that

$$(2.28)\qquad \lim_{k\to\infty} (I - hA^j_{\delta_k, f_k})^{-1} v_k = (I - hA_f^j)^{-1} v \text{ uniformly on } [0,\sigma] \text{ for each } h > 0.$$

PROOF. Setting $u_k = (I - hA^j_{\delta_k, f_k})^{-1} v_k$, we have that $\theta \leqq u_{k+1} \leqq u_k$ by (2.14) in Lemma 2, and hence $u = \lim_{k\to\infty} u_k$ exists both in $\mathcal{L}^1$ and pointwise on $[0,\sigma]$, and so $u \in \mathcal{D}_0$ since $\mathcal{D}_0 \supset \mathcal{D}_{\delta_k}$ and $\mathcal{D}_0$ is closed in $\mathcal{L}^1$. Also, by definition,

$$(2.29)\qquad \phi(u_k)''(x) = \frac{u_k(x) - v_k(x)}{h} - f_k(u_k(x))$$

and since u_k, v_k and $f_k(u_k)$ are uniformly bounded, there is an $M > 0$ such that $|\phi(u_k)''(x)| \leqq M$ for almost all $x \in [0,\sigma]$ and $k \geqq 1$. Thus $\{\phi(u_k)'\}_1^\infty$ is an equicontinuous and uniformly bounded sequence and it follows that $\phi(u)$ is differentiable on $[0,\sigma]$ and that $u_k \to u$ and $\phi(u_k)' \to \phi(u)'$ uniformly on $[0,\sigma]$ as $k \to \infty$. If $\tau = \min\{x \in [0,\sigma] : u(x) = 0\}$ then $u_k(x) \geqq u(x) > 0$ for $x \in [0,\tau)$, and by (2.26), $f_k(u_k(x)) \to f(u(x))$ uniformly on $[0,\rho]$ for each $\rho < \tau$. Thus, by (2.29),

$$\phi(u)''(x) = \frac{u(x) - v(x)}{h} - f(u(x)) \qquad \text{for almost all } x \in [0,\tau)$$

and it follows that if

$$w(x) = \begin{cases} \phi(u)''(x) + f(u(x)) & \text{for } 0 \leqq x < \tau \\ -h^{-1} v(x) & \text{for } \tau \leqq x \leqq \sigma \end{cases}$$

then $u - hw = v$. Since it is clear that $u \in D(A_f^j)$ the proof will be complete once it is shown that $w \in A_f^j u$ [i.e. $(u, w) \in A_f^j$]. To show this it suffices to show that $-h^{-1}v(x) \geqq f(0)$ for almost all $x \in [\tau, \sigma]$. If $\tau \leqq x < y \leqq \sigma$ then

$$\begin{aligned} h\phi(u_k)'(y) - h\phi(u_k)'(x) &= \int_x^y [u_k(r) - v(r)]dr - h\int_x^y f_k(u_k(r))dr \\ &\leqq \int_x^y [u_k(r) - v(r)]dr - h\int_x^y f(u_k(r))dr \end{aligned}$$

where we used that fact that $f \leqq f_{k+1} \leqq f_k$. Letting $k \to \infty$ and using the facts that f is continuous and $u = \phi(u)' = 0$ on $[\tau, \sigma]$ shows that

$$0 \leqq -\int_x^y v(r)dr - h\int_x^y f(0)dr = -\int_x^y v(r)dr - h(y-x)f(0).$$

Thus, $f(0) \leqq -h^{-1}(y-x)^{-1}\int_x^y v(r)dr$ and letting $y \to x+$ shows that $f(0) \leqq -h^{-1}v(x)$ for almost all $x \in [\tau, \sigma]$, and completes the proof of Lemma 5.

COMPLETION OF THE PROOF OF THEOREM 1. Let $f \in \mathcal{F}$ for $k \geqq 1$, let $\eta_k : [0,\infty) \to [0,1]$ be a smooth function such that $\eta_k(0) = 0$ and $\eta_k(u) = 1$ for $u \geqq 1/k$. Assume also that $\eta_k \leqq \eta_{k+1}$ for all k and define f_k on $[0,\infty)$ by $f_k(u) \equiv \eta_k(u)f(u)$ for all $u \geqq 0$ and $k \geqq 1$. If $v \in \mathcal{D}_0$ and we take $\delta_k = 0$ and $v_k = v$ for all $k \geqq 1$, then it is easy to see that each of the suppositions in Lemma 5 is fulfilled. Thus $v \in \mathcal{R}(I - hA_f^j)$ and it follows that $\mathcal{R}(I - hA_f^j) \supset \mathcal{D}(A_f^j) = \mathcal{D}_0$ for all $h > 0$. Since it has already been indicated that (2.23) holds, the Crandall–Liggett theorem implies that the limit in (2.25) exists and if $S_f^j(t)u$ is defined by this limit, then (S1)–(S3) are satisfied. The fact that (2.24) follows from (2.28) and (2.14) in Lemma 2, and (2.24) along with (2.25) show that (S4) is also true. This completes the proof of Theorem 1.

By a combination of Lemma 5 and a theorem of Goldstein [13] (see also Brezis and Pazy [7]) we have the following result for the approximation of the semigroups S_f^j:

PROPOSITION 1. *Suppose that $f \in \mathcal{F}$, $v \in \mathcal{D}_0$, and that δ_k, v_k and f_k are as in Lemma 5. Then*

$$S^j_{\delta_k, f_k}(t)v_k \geqq S_f^j(t)v \qquad \textit{for } k \geqq 1 \tag{2.30}$$

and

$$S_f^j(t)v = \lim_{k\to\infty} S^j_{\delta_k, f_k}(t)v_k \tag{2.31}$$

in $\mathcal{L}^1$, uniformly for t in bounded intervals.

§3. Asymptotic behavior

In this section we continue to use the notations in §2. If $j \in \{D, N\}$, $f \in \mathcal{F}$ and $S_f^j = \{S_f^j(t) : t \geqq 0\}$ is the nonexpansive semigroup on $\mathcal{D}_0$ defined by (2.25), then the function

$$u^j(x,t) \equiv [S_f^j(t)u_0](x), \qquad (x,t) \in [0,\sigma] \times [0,\infty) \tag{3.1}$$

is the generalized solution to the moving boundary problem (2.18). Since $u^j(\cdot,t) \in \mathscr{D}_0$, $u^j(x,t)$ is decreasing in $x \in [0,\sigma]$ and the moving boundary γ^j can be defined directly from u^j by the relation

$$\gamma^j(t) \equiv \inf\{y \in [0,\sigma] : u^j(x,t) = 0 \text{ for all } x \in [y,\sigma]\}.$$

The main purpose of this section is to show that (2.18) has precisely one equilibrium solution in $\mathscr{D}_0$, and that for each $u_0 \in \mathscr{D}_0$, the solution to (2.18) approaches the equilibrium solution as $t \to \infty$. Stated in terms of the semigroup S_f^j we have the following basic result:

THEOREM 2. *Suppose that $j \in \{D,N\}$, that $f \in \mathscr{F}$ with $f(0) \leqq 0$, and that A_f^j is defined by* (2.20) *and S_f^j is defined by* (2.25). *Then there is a unique $w_j^* \in \mathscr{D}^j = D(A_f^j)$ such that*

$$\theta \in A_f^j w_j^* \quad \textit{and} \quad S_f^j(t) w_j^* = w_j^* \qquad \textit{for all } t \geqq 0. \tag{3.2}$$

Also,

$$\lim_{t\to\infty} S_f^j(t)u_0 = w_j^* \textit{ in } \mathscr{L}^1 \textit{ and pointwise on } [0,\sigma] \textit{ for all } u \in \mathscr{D}_0. \tag{3.3}$$

Moreover, if $f(0) < 0$ and

$$\gamma_j^* \equiv \inf\{y \in [0,\sigma] : w_j^*(x) = 0 \textit{ for } y \leqq x \leqq \sigma\}$$

then for each $u_0 \in \mathscr{D}_0$,

$$\lim_{t\to\infty} \gamma^j(t) = \gamma_j^* \tag{3.4}$$

where $\gamma^j(t) \equiv \inf\{y \in [0,\sigma] : [S_f^j(t)u_0](x) = 0$ for $y \leqq x \leqq \sigma\}$.

For the proof of this theorem we use two lemmas, each of which is assumed to be with the suppositions of the theorem.

LEMMA 6. *Suppose that $\theta(x) \equiv 0$ and $c_0(x) \equiv c_0$ for all $x \in [0,\sigma]$. Then*

$$S_f^j(t)\theta \geqq S_f^j(s)\theta \quad \textit{and} \quad S_f^j(t)c_0 \leqq S_f^j(s)c_0 \qquad \textit{for all } t \geqq s \geqq 0 \tag{3.5}$$

and there is a $w_j^* \in \mathscr{D}^j$ such that

$$S_f^j(t)\theta \uparrow w_j^* \downarrow S_f^j(t)c_0 \qquad \textit{as } t \to \infty, \tag{3.6}$$

where the limits are both pointwise and in $\mathscr{L}^1$ on $[0,\sigma]$. Moreover, $S_f^j(t)w_j^* \equiv w_j^*$

for $t \geqq 0$ *and* $\theta \in A_f^j(w_j^*)$. *In particular, there is a* $\gamma_j^* \in [0,\sigma]$ *such that*

$$\begin{aligned}
&\phi(w_j^*)'' + f(w_j^*) = 0 \quad \text{for } 0 < x < \gamma_j^*,\\
(3.7)\qquad &w_j^*(\gamma^*) = 0, \quad \phi(w_j^*)'(\gamma^*) = 0 \quad \text{if } \gamma_j^* < \sigma,\\
&w_j^*(0) = c_0 \quad \text{if } j = D \quad \text{and} \quad \phi(w_j^*)'(0) = 0 \quad \text{if } j = N.
\end{aligned}$$

PROOF. Since $S_f^j(t)$ preserves order on $\mathscr{D}_0$ [see (S4)] and since $\theta \leqq u_0 \leqq c_0$ for all $u_0 \in \mathscr{D}_0$, we see that $S_f^j(t-s)\theta \geqq \theta$, and hence

$$S_f^j(t)\theta = S_f^j(s)S_f^j(t-s)\theta \geqq S_f^j(s)\theta$$

for $t \geqq s \geqq 0$. Similarly, $S_f^j(t-s)c_0 \leqq c_0$ and hence

$$S_f^j(t)c_0 = S_f^j(s)S_f^j(t-s)c_0 \leqq S_f^j(s)c_0$$

for $t \geqq s \geqq 0$. This shows that (3.5) is true and also that there are $z_1^*, z_2^* \in \mathscr{D}_0$ with $z_1^* \leqq z_2^*$ such that

$$\lim_{t\to\infty} S_f^j(t)\theta = z_1^* \quad \text{and} \quad \lim_{t\to\infty} S_f^j(t)c_0 = z_2^*$$

pointwise on $[0,\sigma]$, and hence in $\mathscr{L}^1$ by monotone convergence. By continuity and the semigroup property,

$$S_f^j(t)z_1^* = \lim_{s\to\infty} S_f^j(t)S_f^j(s)\theta = \lim_{s\to\infty} S_f^j(t+s)\theta = z_1^*$$

and hence $S_f^j(t)z_1^* \equiv z_1^*$ for all $t \geqq 0$. Similarly, $S_f^j(t)z_2^* \equiv z_2^*$ for $t \geqq 0$ and this lemma will be established once it is shown that $z_1^* = z_2^*$.

Applying Theorem 1.5, p. 115, of Barbu [5], it follows that $(z_1^*,\theta),(z_2^*,\theta) \in A_f^j$, and so $z_1^*, z_2^* \in \mathscr{D}^j$. Using this fact and setting $\gamma_i^* = \inf\{y \in [0,\sigma] : z_i^*(x) = 0$ for $y \leqq x \leqq \sigma\}$ for $i = 1,2$, shows that the pairs (z_1^*,γ_1^*) and (z_2^*,γ_2^*) are each solutions to the free boundary problem

$$\begin{aligned}
&\phi(w)''(x) + f(w(x)) = 0, \quad x \in [0,\gamma),\\
&w(\gamma) = \phi(w)'(\gamma) = 0 \quad \text{if } \gamma < \sigma,\\
(3.8)\qquad &w(\gamma) = 0 \quad \text{if } \gamma = \sigma,\\
&w(0) = c_0 \quad \text{if } j = D \quad \text{and} \quad \phi(w)'(0) = 0 \quad \text{if } j = N.
\end{aligned}$$

If $j = N$ then clearly $z_1^* \equiv 0$. Since

$$\phi(z_2^*)''(x) = -f(z_2^*(x)) \geqq 0 \qquad \text{for } x \in [0,\gamma_2^*],$$

we see that $\phi(z_2^*)'$ is nondecreasing on $[0,\gamma_2^*]$. Since z_2^* [and hence $\phi(z_2^*)$] is nonincreasing and since $\phi(z_2^*)'(0)=0$ when $j=N$, we immediately conclude that $\gamma_2^*=0$. Thus $z_1^*\equiv z_2^*\equiv 0$ when $j=N$. Now suppose that $j=D$. Since $z_1^*\leqq z_2^*$ we have that $\gamma_1^*\leqq\gamma_2^*$, and hence both z_1^* and z_2^* satisfy the differential equation in (3.8) for $x\in[0,\gamma_1^*]$. From this it follows that

$$[\phi(z_2^*)-\phi(z_1^*)]''\cdot[\phi(z_2^*)-\phi(z_1^*)]=-[f(z_2^*)-f(z_1^*)]\cdot[\phi(z_2^*)-\phi(z_1^*)]$$

for all $x\in(0,\gamma_1^*)$. Since ϕ is increasing on $[0,\infty)$ and f is nonincreasing, the right side of this equation is nonnegative, and so

$$\int_0^{\gamma_1^*}[\phi(z_2^*)-\phi(z_1^*)]''\cdot[\phi(z_2^*)-\phi(z_1^*)]dx\geqq 0.$$

Integrating by parts and using the boundary conditions

$$\phi(z_2^*)(0)=\phi(z_1^*)(0)=\phi(c_0)\quad\text{and}\quad\phi(z_1^*)(\gamma_1^*)=\phi(z_1^*)'(\gamma_1^*)=0,$$

it follows that

$$\int_0^{\gamma_1^*}\{[\phi(z_2^*)-\phi(z_1^*)]'\}^2\leqq\phi(z_2^*)'(\gamma_1^*)\phi(z_2^*)(\gamma_1^*).$$

Since $x\to\phi(z_2^*)(x)$ is nonnegative and nonincreasing, we have that $\phi(z_2^*)'(\gamma_1^*)\phi(z_2^*)(\gamma_1^*)\leqq 0$, and hence $\phi(z_2^*)'(x)\equiv\phi(z_1^*)'(x)$ for all $x\in[0,\gamma_1^*]$. Since $\phi(z_2^*)(0)=\phi(z_1^*(0))=\phi(c_0)$ when $j=D$ we conclude that $\phi(z_2^*)(x)\equiv\phi(z_1^*)(x)$ on $[0,\gamma_1^*]$. From this it follows that $z_2^*=z_1^*$ on $[0,\sigma]$ and the proof of Lemma 6 is complete.

LEMMA 7. *Suppose that $w_j^*\in\mathscr{D}^j$ and γ^* are as in Lemma 6 and that $u_0\in\mathscr{D}_0$ with $u_0\geqq w_j^*$. If $\gamma_j^*<\sigma$ then*

$$(3.9)\quad |S_f^j(t)u_0-w_j^*|_1\leqq|u_0-w_j^*|_1+\int_0^t\int_{\gamma_j^*}^{\gamma^j(\tau)}f([S_f^j(\tau)u_0](x))dxd\tau\qquad\text{for all } t>0$$

where

$$\gamma^j(t)\equiv\inf\{y\in[0,\sigma]:[S_f^j(t)u_0](x)=0\ \text{for}\ y\leqq x\leqq\sigma\}\qquad\text{for each } t>0.$$

PROOF. For each $k\geqq 1$ let η_k be a smooth, nondecreasing function on $[1/k,\infty)$ such that $\eta_k(1/k)=0$ and $\eta_k(u)=1$ for $u\geqq 2/k$. Assume also that $\eta_{k+1}\geqq\eta_k$ on $[1/k,\infty)$ for $k\geqq 1$ and define $f_k(u)\equiv\eta_k(u)f(u)$ for $u\geqq 0$. Then $f_k(1/k)=0$ and by appropriately defining $f_k(u)$ for $u<1/k$, we may assume that $f_k\in\mathscr{F}_{1/k}$ and $f_{k+1}\leqq f_k$ for all $k\geqq 1$. Now set

$$\mathscr{E}_k=\{v\in D(A^j_{1/k,f_k}): v\text{ is infinitely differentiable and } v\geqq w^*\}$$

and let $v \in \mathscr{E}_k$. Then the function

$$u(x,t) \equiv [S^j_{1/k,f_k}(t)v](x) \qquad \text{for } t \geqq 0, \quad 0 \leqq x \leqq \sigma$$

is a classical solution to

$$\begin{aligned} &u_t = \phi(u)_{xx} + f_k(u), \qquad t>0, \quad 0<x<\sigma, \\ &u_k(\sigma,t) = 1/k, \\ (3.10) \qquad &u_k(0,t) = c_0 \quad \text{if } j = D \quad \text{and} \quad \partial_x\phi(u_k(0,t)) = 0 \quad \text{if } j = N, \\ &u_k(x,0) = v(x), \quad 0<x<\sigma \end{aligned}$$

(see Lemma 4 in §2). Also, by Proposition 1 in §2 and property (S4), the facts that $v \geqq w^*$ and $f_k \geqq f$ imply that

$$S^j_{1/k,f}(t)v \geqq S^j_f(t)v \geqq S^j_f(t)w^* = w^*,$$

and so $u(x,t) \geqq w^*(x)$ for all $t \geqq 0$, $0 \leqq x \leqq \sigma$. Since w^* satisfies (3.7) in Lemma 6 and since u satisfies (3.10), if

$$p(t) \equiv \int_0^\sigma [u(x,t) - w^*(x)]dx \qquad \text{for } t \geqq 0,$$

then, suppressing the variables,

$$\begin{aligned} p'(t) &= \int_0^{\gamma^*} u_t dx + \int_{\gamma^*}^\sigma u_t dx \\ &= \int_0^{\gamma^*} [\phi(u)_{xx} + f_k(u) - \phi(w^*)_{xx} - f(w^*)]dx + \int_{\gamma^*}^\sigma [\phi(u)_{xx} + f_k(u)]dx \\ &= \int_0^{\gamma^*} [\phi(u)_{xx} - \phi(w^*)_{xx}]dx + \int_0^{\gamma^*} [f_k(u) - f(w^*)]dx \\ &\quad + \int_{\gamma^*}^\sigma \phi(u)_{xx}dx + \int_{\gamma^*}^\sigma f_k(u)dx \\ &= [\phi(u)_x - \phi(w^*)_x]_{x=0}^{x=\gamma^*} + [\phi(u)_x]_{\gamma^*}^\sigma \\ &\quad + \int_0^{\gamma^*} [f_k(u) - f(w^*)]dx + \int_{\gamma^*}^\sigma f_k(u)dx \\ &= -[\phi(u)_x(0) - \phi(w^*)_x(0)] \\ &\quad + \phi(u)_x(\sigma) + \int_0^{\gamma^*} [f_k(u) - f(w^*)]dx + \int_{\gamma^*}^\sigma f_k(u)dx. \end{aligned}$$

Since $u \geqq w^*$ and $u(0,t) = w^*(0)$ if $j = D$, it is easy to see that

$$\phi(u)_x(0) - \phi(w^*)_x(0) \geqq 0.$$

Since $\phi(u)$ is nonincreasing, $\phi(u)_x(\sigma) \leqq 0$ and it follows that

$$p'(t) \leqq \int_0^{\gamma^*} [f_k(u) - f(w^*)]dx + \int_{\gamma^*}^{\sigma} f_k(u)dx.$$

Integrating each side of this inequality from 0 to t,

$$p(t) - p(0) \leqq \int_0^t \int_0^{\gamma^*} [f_k(u) - f(w^*)] + \int_0^t \int_{\gamma^*}^{\sigma} f_k(u)$$

and it follows that if $v \in \mathscr{E}_k$ and u is the solution to (3.10), then

$$|u(\cdot,t) - w^*|_1 \leqq |v - w^*|_1 + \int_0^t \int_0^{\gamma^*} [f_k(u) - f(w^*)]dxd\tau + \int_0^t \int_{\gamma^*}^{\gamma(\tau)} f_k(u)dxd\tau$$

where we use the fact that $f_k(u) \leqq 0$ in estimating the last inegral. Since $u(\cdot,t) = S^j_{1/k,f_k}(t)v$ and $S^j_{1/k,f_k}(t)$ is nonexpansive for the $\mathscr{L}^1$ norm in v [see property (S3)] and continuous for the $\mathscr{L}^1$ norm in t and since both f and f_k are continuous on $\mathscr{D}_{1/k}$ relative to the $\mathscr{L}^1$ norm on $[0,\sigma]$, it follows that

$$\begin{aligned}|S^j_{1/k,f_k}(t)v - w^*|_1 \leqq |v - w^*|_1 &+ \int_0^t \int_0^{\gamma^*} [f_k(S^j_{1/k,f_k}(t)v) - f(w^*)]dxd\tau \\ &+ \int_0^t \int_{\gamma^*}^{\gamma(\tau)} f_k(S^j_{1/k,f_k}(\tau)v)dxd\tau\end{aligned}$$

for all $v \in \mathscr{D}_{1/k}$ with $v \geqq w^*$. (Note that the $\mathscr{L}^1$ closure of $\mathscr{E}_k$ equals $\{v \in \mathscr{D}_{1/k} : v \geqq w^*\}$.) Therefore, we can select a sequence $\{v_k\}^\infty$, such that $v_k \in \mathscr{D}_{1/k}$, $v_k \geqq u_0$, $v_{k+1} \leqq v_k$ and $\mathscr{L}^1\text{-}\lim_{k\to\infty} v_k = u_0$. If $u_k(x,t) \equiv [S^j_{1/k,f_k}(t)v_k](x)$ and $u(x,t) \equiv [S^j_f(t)u_0](x)$, then

$$u_k(x,t) \geqq u(x,t) \quad \text{and} \quad \mathscr{L}^1\text{-}\lim_{k\to\infty} u_k(\cdot,t) = u(\cdot,t)$$

by Proposition 1 in §2. Thus if $0 < x < \gamma^*$, then $x < \gamma(t)$ and $u_k(x,t) \geqq u(x,t) > 0$, and so

$$f_k(u_k(x,t)) = f(u_k(x,t)) \qquad \text{for } t > 0, \quad 0 < x < \gamma^*$$

when k is sufficiently large. From this it follows that $f_k(u_k(x,t)) \to f(u(x,t))$ pointwise almost everywhere on $(0,t) \times (0,\gamma^*)$. Setting $v = v_k$ in (3.11) and letting $k \to \infty$ shows that

$$|S_f^j(t)u_0 - w^*|_1 \leqq |u_0 - w^*|_1 + \int_0^t \int_0^{\gamma^*} [f(S_f^j(\tau)u_0) - f(w^*)]dxd\tau + \int_0^t \int_{\gamma^*}^{\gamma(\tau)} f(S_f^j(t)u_0)dxd\tau$$

for each $t > 0$. Since f is nonincreasing and $S_f^j(t)u_0 \geqq w^*$, the first integral term on the right side of this inequality is nonpositive, and we see that (3.9) is true.

PROOF OF THEOREM 2. Lemma 6 shows that (3.2) is true. Also, since $\theta \leqq u_0 \leqq c_0$ for all $u_0 \in \mathscr{D}_0$, we have that $S_f^j(t)\theta \leqq S_f^j(t)u_0 \leqq S_f^j(t)c_0$ for all $t \geqq 0$ by (S4), and so assertions (3.3) and (3.4) will be established once they are shown to be true for $u_0 = \theta$ and $u_0 = c_0$. Thus by (3.6) in Lemma 6 we also see that (3.3) in Theorem 2 is satisfied. Let γ^* and $\gamma(t)$ be as in Theorem 2 with $u_0 = \theta$. Thus $\gamma(t) \leqq \gamma^*$ for all $t \geqq 0$ since $S_f^j(t)\theta \leqq w^*$ and $\gamma(t)$ is nondecreasing by (3.5) in Lemma 6. Moreover, if $0 \leqq y < \gamma^*$ then $[S_f^j(t)\theta](y) \uparrow w^*(y) > 0$ as $t \to \infty$, and so there is a $T(y) > 0$ such that $[S_f^j(t)\theta](y) > 0$ for $t \geqq T(y)$. Thus $\gamma(t) \geqq y$ for all $t \geqq T(y)$ and it follows that $\gamma(t) \uparrow \gamma^*$ as $t \to \infty$. Now suppose that $u_0 = c_0$ and $\gamma(t)$ is the corresponding moving boundary. Then (3.5) shows that $\gamma(t) \geqq \gamma^*$ and $\gamma(t)$ is nonincreasing. Thus let $\lambda \equiv \lim_{t\to\infty} \gamma(t)$. Then $\lambda \geqq \gamma^*$ and since $f(0) < 0$ and f is nonincreasing, it follows from (3.9) in Lemma 7 that

$$|S_f^j(t)c_0 - w^*|_1 \leqq |c_0 - w^*|_1 + \int_0^t \int_{\gamma^*}^{\gamma(\tau)} f([S_f^j(\tau)c_0](x))dxd\tau$$

$$\leqq |c_0 - w^*|_1 + \int_0^t \int_{\gamma^*}^{\gamma(\tau)} f(0)dxd\tau \leqq |c_0 - w^*|_1 + t(\lambda - \gamma^*)f(0).$$

Letting $t \to \infty$ shows that $\lambda = \gamma^*$ and completes the proof of the Theorem.

The original physical model motivating the study of these equations is the diffusion of oxygen in absorbing tissue. In this case the solution $u(x, t)$ is the concentration of the oxygen and moving boundary $\gamma(t)$ is the penetration depth of the oxygen into the tissue. Initially there is no concentration of oxygen in the tissue and the oxygen concentration is held equal to the constant c_0 at the surface [this corresponds to the Dirichlet boundary condition at $x = 0$ and $j = D$ in equation (2.18)]. Once the penetration depth $\gamma^D(t)$ has approximately reached equilibrium state [see (3.4) in Theorem 2 with $j = D$] the surface is sealed (this corresponds to the Neumann boundary condition at $x = 0$ and $j = N$ in equation (2.18)] and the penetration depth $\gamma^N(t)$ then reaches back to the surface [see (3.4) in Theorem 2 with $j = N$]. Therefore, the two principal equations to be analyzed are the system

$$
\begin{aligned}
&u_t^D(x,t)=\phi(u^D(x,t))_{xx}+f(u^D(x,t)), \quad t>0, \quad 0<x<\gamma^D(t), \\
(3.12) \qquad &u^D(0,t)=c_0, \quad u^D(\gamma^D(t),t)=\phi(u^D(\gamma^D(t),t))_x=0, \quad t>0, \\
&u^D(x,0)=0 \quad \text{for } 0<x<\sigma,
\end{aligned}
$$

and the system

$$
\begin{aligned}
&u_t^N(x,t)=\phi(u^N(x,t))_{xx}+f(u^N(x,t)), \quad t>0, \quad 0<x<\gamma^D(t), \\
(3.13) \qquad &\phi(u^D(0,t))_x=0, \quad u^N(\gamma^N(t),t)=\phi(u^N(\gamma^N(t),t))_x=0, \quad t>0, \\
&u^N(x,0)=w_D^*(x), \quad 0<x<0,
\end{aligned}
$$

where w_D^* is as in Theorem 2:

$$
\begin{aligned}
&\phi(w_D^*)''(x)+f(w_D^*(x))=0, \quad 0<x<\gamma_D^*, \\
(3.14) \qquad &w_D^*(0)=c_0, \quad w_D(\gamma_D^*)=\phi(w_D^*)'(\gamma_D^*)=0, \\
&w_D^*(x)=0 \quad \text{for } \gamma_D^*<x\leqq\sigma.
\end{aligned}
$$

We assume now that $f(0)<0$ and $\gamma_D^*<\sigma$.

According to Theorem 2, we know that $u^D(x,t)\uparrow w_D^*(x)$ and $\gamma^D(t)\uparrow\gamma_D^*$ as $t\to\infty$, and that $u^N(x,t)\to 0$ and $\gamma^N(t)\to 0$ as $t\to\infty$. However, our next result shows that the precise asymptotic behavior in these two cases is considerably different.

THEOREM 3. *Suppose that ϕ is $\mathscr{C}^\infty$ on $(0,\infty)$ and $g(\xi)\equiv f(\phi^{-1}(\xi))$ is $\mathscr{C}^{1+\alpha}$ on $[0,\infty)$ for some $\alpha>0$. Suppose also that $f(0)<0$, w_D^* is the solution to* (3.14) *with $\gamma_D^*<\sigma$, u^D,γ^D is the solution to* (3.12), *and u^N,γ^N is the solution to* (3.13). *Then*

(i) $u^D(x,t)\uparrow w_D^*(x)$, $\gamma^D(t)\uparrow\gamma_D^*$ *as* $t\to\infty$, $\gamma^D(t)<\gamma_D^*$, *and* $u^D(x,t)<w_D^*(x)$ *for all* $t>0$, $x\in(0,\gamma_D^*)$; *and*

(ii) $u^N(x,t)\downarrow 0$, $\gamma^N(t)\downarrow 0$ *as* $t\to\infty$, *and there is a finite* $\tau>0$ *such that* $u^N(x,t)\equiv 0$, $\gamma^N(t)\equiv 0$ *for all* $t\geqq\tau$, $0\leqq x\leqq\sigma$.

The hypothesis that ϕ is $\mathscr{C}^\infty$ on $(0,\infty)$ is not very restrictive, since ϕ is normally of the form $\phi(u)\equiv du^m$ where $d>0$ and $m\geqq 1$. The function $g(\xi)\equiv f(\phi^{-1}(\xi))$ is required to be $\mathscr{C}^{1+\alpha}$ on $[0,\infty)$ so that the results in [19] can be applied. Note that if $\phi(u)\equiv du^m$ and $f(\xi)\equiv b\xi^k$, then $g(\xi)=f(\phi^{-1}(\xi))$ is $\mathscr{C}^{1+\alpha}$ on $[0,\infty)$ if $k=0$ or if $k\geqq m$. Of course, the main point in Theorem 3 is that the moving boundary in (3.12) remains strictly less than its limiting value for all time, whereas the moving boundary in (3.13) actually equals its limiting value after a finite amount of time. For the proof of Theorem 3 we use several preliminary lemmas, each of which is assumed to be under the hypotheses of the theorem.

LEMMA 8. *Suppose that* $\delta \geqq 0$, $f_\delta \in \mathcal{F}_\delta$ *and* $z_0 \in \mathcal{D}_\delta$ *is such that* $\phi(z_0)$ *is twice continuously differentiable on* $[0,\sigma]$. *If*

$$\begin{aligned} &\phi(z_0)'' + f_\delta(z_0) \geqq 0 \quad \text{for } z \in [0,\sigma], \\ &z_0(0) \leqq c_0 \quad \text{and} \quad z_0(\sigma) = \delta, \end{aligned} \tag{3.15}$$

then

$$S^D_{\delta,f_\delta}(t)z_0 \geqq S^D_{\delta,f_\delta}(s)z_0 \geqq z_0 \qquad \text{for all } t \geqq s \geqq 0.$$

PROOF. Setting $v_0 = (I - hA^D_{\delta,f_\delta})^{-1}z_0$ we see that

$$v_0 - h[\phi(v_0)'' + f_\delta(v_0)] = z_0 \geqq z_0 - h[\phi(z_0)'' + f(z_0)]$$

and hence that

$$v_0 - z_0 \geqq h[\phi(v_0) - \phi(z_0)]'' + h[f(v_0) - f(z_0)]. \tag{3.16}$$

Let $x_0 \in [0,\sigma]$ be such that

$$\phi(v_0(x_0)) - \phi(z_0(x_0)) = \min\{\phi(v_0(x)) - \phi(z_0(x)) : x \in [0,\sigma]\}.$$

Since $v_0(0) = c_0 \geqq z_0(x_0)$ and $v_0(\sigma) = \delta = z_0(\sigma)$ we see that a negative minimum for $\phi(v_0) - \phi(z_0)$ (and hence for $v_0 - z_0$) can occur only if $x_0 \in (0,\sigma)$. Since this would imply that $[\phi(v_0) - \phi(z_0)]''(x_0) \geqq 0$ and since f_δ is nonincreasing, it is impossible for (3.16) to hold at such an x_0, and we conclude that $\phi(v_0) \geqq \phi(z_0)$, and hence $v_0 \geqq z_0$. Thus $(I - hA^D_{\delta,f_\delta})^{-1}z_0 \geqq z_0$ and the order-preserving property (2.14) in Lemma 2 shows that

$$(I - hA^D_{\delta,f_\delta})^{-n}z_0 \geqq (I - hA^D_{\delta,f_\delta})^{-n+1}z_0 \geqq \cdots \geqq (I - hA^D_{\delta,f_\delta})^{-1}z_0 \geqq z_0$$

for all $h > 0$ and positive integers n. Setting $h = t/n$ and letting $n \to \infty$ implies that $S^D_{\delta,f_\delta}(t)z_0 \geqq z_0$ for all $t \geqq 0$ by (2.15). Finally, using the order-preserving property (S4), if $t \geqq s \geqq 0$ then

$$S^D_{\delta,f_\delta}(t)z_0 = S^D_{\delta,f_\delta}(s)S^D_{\delta,f_\delta}(t-s)z_0 \geqq S^D_{\delta,f_\delta}(s)z_0$$

and the proof of this lemma is complete.

LEMMA 9. *Suppose that* $\{\delta_k\}_1^\infty$ *is a decreasing sequence in* $(0,c_0)$ *with* $\lim_{k\to\infty}\delta_k = 0$ *and that* $\{f_k\}_1^\infty$ *is a sequence of* $\mathcal{C}^\infty$ *functions with* $f_k \in \mathcal{F}_{\delta_k}$, $f_{k+1} \leqq f_k$ *for* $k \geqq 1$, *and* $\lim_{k\to\infty} f_k(\xi) = f(\xi)$ *uniformly on* $[\eta, c_0]$ *for each* $\eta > 0$. *For each*

$k \geqq 1$ let v^k be the solution to the semilinear parabolic equation

$$\begin{aligned} &v_t^k = v_{xx}^k + f_k(\phi^{-1}(v^k)), \quad t>0, \quad 0<x<\sigma, \\ (3.17) \qquad &v^k(0,t) = \phi(c_0), \quad v^k(\sigma,t) = \phi(\delta_k), \quad t>0, \\ &v^k(x,0) = \phi(\delta_k), \quad 0<x<\sigma, \end{aligned}$$

then $\phi(\delta_k) \leqq v^k(x,t) \leqq \phi(c_0)$ for all $(x,t) \in [0,\sigma]\times[0,\infty)$ and v^k is $\mathscr{C}^\infty$ in both x and t on $[0,\sigma]\times(0,\infty)$. Moreover, $v_t^k(x,t) \geqq 0$ for all $(x,t)\in[0,\sigma]\times(0,\infty)$ and $\mathscr{L}^1$-$\lim_{t\to 0+} v^k(\cdot,t) = \phi(\delta_k)$ for each $k \geqq 1$.

Proof. The maximum principle shows that $\phi(\delta_k) \leqq v^k \leqq \phi(c_0)$ and since f_k and ϕ^{-1} are $\mathscr{C}^\infty$ on $[\phi(\delta_k), \phi(c_0)]$ and (3.17) is a semilinear parabolic equation we also have that v^k is $\mathscr{C}^\infty$ on $[0,\sigma]\times(0,\infty)$. Noting that

$$v_{xx}^k(x,0) + f_k(\phi^{-1}(v^k(x,0))) = f_k(\phi^{-1}(\phi(\delta_k))) = f_k(\delta_k) = 0$$

since $f_k \in \mathscr{F}_{\delta_k}$, we see from the preceding lemma (with $\phi(u)\equiv u$) that $v_t^k \geqq 0$. Since (3.17) generates a C_0-semigroup in $\mathscr{L}^1$ we also have that $v^k(\cdot,t) \to v^k(\cdot,0) \equiv \phi(\delta_k)$ is $\mathscr{L}^1$ as $t\to 0+$ and the proof is complete.

Lemma 10. *Suppose that $\mu \equiv \max\{\phi'(\xi) : 0 \leqq \xi \leqq c_0\}$ and that v is the solution to the semilinear moving boundary problem*

$$\begin{aligned} &v_t = \mu v_{xx} + \mu f(\phi^{-1}(v)), \quad t>0, \quad 0<x<\lambda(t), \\ (3.18) \qquad &v(0,t) = \phi(c_0), \quad v(\lambda(t),t) = v_x(\lambda(t),t) = 0, \quad t>0, \\ &v(x,0) = 0. \end{aligned}$$

Then the solution u^D to (3.12) *satisfies*

$$(3.19) \qquad \phi(u^D(k,t)) \leqq v(x,t) \qquad \textit{for all } t \geqq 0, \quad x \in [0,\sigma].$$

Proof. For each $k \geqq 1$ let v^k be the solution to (3.17) and let $\varepsilon_k > 0$ be such that $|v^k(\cdot,\varepsilon_k)|_1 \leqq 2\sigma\phi(\delta_k)$. We know from Proposition 1 that $v^k \to v$ as $k\to\infty$, uniformly on $[0,\sigma]\times[0,T]$ for each $T>0$ (see also Lemma 1 and equation (2.14) in [19]). If $u^k \equiv \phi^{-1}(v^k(\cdot,\varepsilon_k))$ then u^k is $\mathscr{C}^\infty$ and in $D(A^D_{\delta_k,f_k})$. Thus, by Lemma 4, $u^k(x,t) \equiv [S^D_{\delta_k,f_k}(t)u_0^k](x)$ is a classical solution to

$$\begin{aligned} &u_t^k = \phi(u^k)_{xx} + f_k(u^k), \quad t>0, \quad 0<x<\sigma, \\ (3.20) \qquad &u^k(0,t) = c_0, \quad u^k(\sigma,t) = \delta_k, \quad t>0, \\ &u^k(x,0) = \phi^{-1}(v^k(x,\varepsilon_k)), \quad 0<x<\sigma. \end{aligned}$$

Since $v_t^k(x,\varepsilon_k) \geqq 0$ by Lemma 9, we see that

$$\phi(u^k(x,0))_{xx} + f_k(u^k(x,0)) = v_{xx}^k(x,\varepsilon_k) + f_k(\phi^{-1}(v^k(x,\varepsilon_k))) \geqq 0$$

and it follows from Lemma 8 that $u_t^k \geqq 0$ on $[0,\sigma] \times (0,\infty)$. Therefore, if $w^k(x,t) \equiv \phi(u^k(x,t))$, then

$$w_t^k = \phi'(u^k)u_t^k \leqq \mu u_t^k = \mu[\phi(u^k)_{xx} + f_k(u^k)] = \mu w_{xx}^k + \mu f_k(\phi^{-1}(w^k))$$

and it follows from the maximum principle that $w^k \leqq z^k$ where z^k is the solution to

$$z_t^k = \mu z_{xx}^k + \mu f_k(\phi^{-1}(z^k)),$$
$$z^k(0,t) = \phi(c_0), \quad z^k(\sigma,t) = \delta_k,$$
$$z^k(x,0) = v^k(x,\varepsilon_k).$$

Therefore, $\phi(u^k(x,t)) \leqq z^k(x,t)$ for all $t \geqq 0$, $x \in [0,\sigma]$, and since $u^k \to u^D$ and $z^k \to v$ as $k \to \infty$, we see that Lemma 10 is true.

Proof of (i) in Theorem 3. Since $z_D^*(x) \equiv \phi(w_D^*(x))$, $\lambda_D^* = \gamma_D^*$ is the equilibrium solution to (3.18), we have from Theorem 3 and Proposition 3 in [19] that the solution v,λ to (3.18) satisfies $v(x,t) < z_D^*(x)$ and $\lambda(t) < \gamma_D^*$ for all $t \geqq 0$, $x \in [0,\gamma^*)$. Thus, using (3.19),

$$u^D(x,t) \leqq \phi^{-1}(v(x,t)) < \phi^{-1}(z_D^*(x)) = w_D^*(x)$$

and $\gamma^D(t) \leqq \lambda(t) < \gamma_D^*$ for all $t \geqq 0$ and $x \in (0,\gamma_D^*)$. This completes the proof of part (i) of Theorem 3.

Lemma 11. *Suppose that $\delta \geqq 0$, $f_\delta \in \mathscr{F}_\delta$ and $z_0 \in \mathscr{D}_\delta$ is such that $\phi(z_0)$ is twice continuously differentiable on $[0,\sigma]$. If*

$$\begin{aligned} &\phi(z_0)'' + f_\delta(z_0) \leqq 0 \qquad \text{for } x \in [0,\sigma], \\ &\phi(z_0)'(0) \leqq 0 \quad \text{and} \quad z_0(\sigma) \geqq \delta, \end{aligned} \tag{3.21}$$

then

$$S_{\delta,f_\delta}^N(t)z_0 \leqq S_{\delta,f_\delta}^N(s)z_0 \leqq z_0 \qquad \text{for all } t \geqq s \geqq 0.$$

Indication of Proof. Following the ideas in the proof of Lemma 8, if $v_0 = (I - hA_{\delta,f_\delta}^N)^{-1}z_0$ then

$$v_0 - z_0 \leqq h[\phi(v_0) - \phi(z_0)]'' + h[f(v_0) - f(z_0)].$$

Now let $x_0 \in [0,\sigma]$ be such that

$$\phi(v_0(x_0)) - \phi(z_0(x_0)) = \max\{\phi(v_0(x)) - \phi(z_0(x)) : x \in [0,\sigma]\}.$$

If this maximum is positive then $x_0 \in [0,\sigma)$ and if $x_0 = 0$ then $\phi(v_0)'(0) - \phi(z_0)'(0) \geqq 0$ and it follows that $[\phi(v_0) - \phi(z_0)]''(x_0) \leqq 0$. This is impossible since f_δ is nonincreasing and we conclude that $(I - hA^N_{\delta,f_\delta})^{-1} z_0 \leqq z_0$ for all $h \geqq 0$. The assertions now follow analogously to the proof of Lemma 8.

Now suppose that $\{\delta_k\}_1^\infty$, and $\{f_k\}_1^\infty$, are as in Lemma 9 and for each $k \geqq 1$ let y_k be the (unique) solution to the boundary value problem

$$\begin{aligned} &y_k''(x) + f_k(\phi^{-1}(y_k(x))), \quad 0 < x < \gamma^*, \\ &y_k(0) = \phi(c_0), \quad y_k(\gamma^*) = \phi(\delta_k). \end{aligned} \tag{3.22}$$

Then the maximum principle implies that $\delta_k \leqq y_k(x) \leqq c_0$ for all $x \in [0,\gamma^*]$, and hence that $y_k'(0) \leqq 0$ (in fact, $y_k \in \mathscr{D}_{\delta_k}$ with $\sigma = \gamma^*$). By Lemma 11 with $\phi(u) \equiv u$ the solution z^k to

$$\begin{aligned} &z_t^k = z_{xx}^k + f_k(\phi^{-1}(z^k)), \quad t > 0, \quad 0 < x < \gamma^*, \\ &z_x^k(0,t) = 0, \quad z^k(\gamma^*,t) = \phi(\delta_k), \quad t > 0, \\ &z^k(x,0) = y_k(x), \quad 0 < x < \gamma^* \end{aligned} \tag{3.23}$$

satisfies $z_t^k(x,t) \leqq 0$ for all $(x,t) \in [0,\gamma^*] \times (0,\infty)$. Since (3.23) is a semilinear equation and $f_k(\phi^{-1}(\cdot))$ is $\mathscr{C}^\infty$, the solution z^k is $\mathscr{C}^\infty$ in (x,t) for $t > 0$ and also satisfies $z^k(\cdot,t) \to y_k$ in $\mathscr{L}^1$ as $t \to 0+$. Moreover, the solution y_k to (3.22) satisfies $y_k \to \phi(w_D^*)$ in $\mathscr{L}^1$ as $k \to \infty$, and it follows that positive numbers ε_k can be selected so that

$$\begin{aligned} &\phi^{-1}(z^k(\cdot,\varepsilon_k)) \in D(A^N_{\delta_k,f_k}) \text{ and is } \mathscr{C}^\infty, \\ &z_t^k(x,\varepsilon_k) \leqq 0, \quad \text{and } |z^k(\cdot,\varepsilon_k) - w_D^*|_1 \to 0 \quad \text{as } k \to \infty. \end{aligned} \tag{3.24}$$

LEMMA 12. *For each $k \geqq 1$ let v^k be the solution to*

$$\begin{aligned} &v_t^k = \phi(v^k)_{xx} + f_k(v^k), \quad t > 0, \quad 0 < x < \gamma^*, \\ &\phi(v^k)_x(0,t) = 0, \quad v^k(\gamma^*,t) = \delta_k, \quad t > 0, \\ &v^k(x,0) = \phi^{-1}(z^k(x,\varepsilon_k)), \quad 0 < x < \gamma^*. \end{aligned} \tag{3.25}$$

Then v^k is $\mathscr{C}^2$ in x and $\mathscr{C}^1$ in t on $[0,\sigma] \times [0,\infty)$, $v_t^k(x,t) \leqq 0$ for all $t > 0$, and

$$u^N(\cdot,t) = \mathscr{L}^1\text{-}\lim_{k\to\infty} v^k(\cdot,t) \tag{3.26}$$

uniformly for t in bounded subsets of $[0,\infty)$.

PROOF. If $v_0^k(x) \equiv \phi^{-1}(z^k(x,\varepsilon_k))$ for $0 \leqq x \leqq \gamma^*$, then v_0^k is $\mathscr{C}^\infty$ and in $D(A^N_{\delta_k,f_k})$ by (3.24), and since

$$v^k(x,t) = [S^N_{\delta_k,f_k}(t)v_0^k](x) \qquad \text{for } (x,t) \in [0,\gamma^*] \times [0,\infty),$$

it follows from Lemma 4 (with $\sigma = \gamma^*$) that v^k is $\mathscr{C}^2$ in x and $\mathscr{C}^1$ in t. Since

$$\phi^{-1}(z^k(\cdot,\varepsilon_k)) \to \phi^{-1}(\phi(w_D^*)) = w_D^*$$

in $\mathscr{L}^1$ as $k \to \infty$, we may apply the proof techniques in Lemma 5 to show that $v^k(\cdot,t) \to u^N(\cdot,t)$ is $\mathscr{L}^1$, uniformly for t bounded. Notice that the convergence is not monotone; however, (2.29) in the proof of Lemma 5 is still valid and the proof follows analogously using the injectiveness of $(I - hA_f^N)$. Since $z_t^k(x,\varepsilon_k) \leqq 0$ we have from Lemma 11 that

$$S^N_{\delta_k,f_k}(t)v_0^k \leqq S_{\delta_k,f_k}(s)v_0^k \leqq v_0^k \qquad \text{for all } t \geqq s \geqq 0,$$

and so $v_t^k(x,t) \leqq 0$ for all $t > 0$. This completes the proof of Lemma 12.

PROOF OF (ii) IN THEOREM 3. Since $v_t^k(x,t) \leqq 0$, it follows from (3.26) that $u^N(x,t)$ is nonincreasing in t and hence $\gamma^N(t)$ is nonincreasing for $t \geqq 0$. This shows that $\gamma^N(t) \downarrow 0$ as $t \to \infty$. Since the solution v^k is nonincreasing in x (in fact, $v^k(\cdot,t) \in \mathscr{D}_{\delta_k}$) we have that $\phi(v^k(x,t))$ is nonincreasing in x, and since

$$\frac{\partial}{\partial x}[\phi(v^k(0,t))] = 0$$

by the boundary conditions in (3.25), we see that

$$\frac{\partial^2}{\partial x^2}[\phi(v^k(0,t))] \leqq 0.$$

From the differential equation in (3.25) it now follows that

$$v_t^k(0,t) \leqq f_k(v^k(0,t)) \qquad \text{for all } t > 0, \quad k \geqq 1.$$

From this it follows that $v^k(0,t) \leqq \nu_k(t)$ for all $t \geqq 0$, where ν_k is the solution to the ordinary differential equation

$$\nu_k'(t) = f_k(\nu_k(t)), \quad \nu_k(0) = c_0, \quad t \geqq 0.$$

Since v^k is nonincreasing in x,

$$v^k(x,t) \leqq \nu_k(t) \qquad \text{for all } t \geqq 0, \quad x \in [0,\gamma^*]. \tag{3.27}$$

Note that if ν is the solution to

$$\nu'(t) = f(\nu(t)), \quad \nu(0) = c_0, \quad t \geqq 0,$$

then $\nu'(t) \leqq f(0) < 0$ for as long as $\nu(t) > 0$, and it follows that there is a $t_1 > 0$ such that $\nu(t) > 0$ for $t < t_1$ and $\nu(t) \leqq 0$ for $t \geqq t_1$. Then $\nu_k(t) \to \nu(t)$ for all $t \in [0, t_1]$ as $k \to \infty$, and it follows from (3.27) that $u^N(x,t) \leqq \nu(t)$ for all $(x,t) \in [0, \gamma^*] \times [0, t_1]$. Since $\nu(t_1) = 0$ the proof of Theorem 3 is complete.

§4. Regularity of solutions in a special case

The existence results obtained in §2 are of a very general nature and the purpose of this section is to indicate that in certain situations one can establish additional continuity and differentiability properties of the semigroups generated by this moving boundary problem. In the semilinear case [i.e., the case when $\phi(u) \equiv du$], it is shown in [19] that the solution $u^j = u^j(x,t)$ to (2.18) is $\mathscr{C}^1$ in t and $\mathscr{C}^2$ in x so long as $u^j(x,t) > 0$, and that it is $\mathscr{C}^\nu$ in t and $\mathscr{C}^{1+\nu}$ in x on $[0,\sigma] \times (0,\infty)$ for some $\nu > 0$. However, the solution always has a singular point in its t partial derivative (and hence in its second x partial derivative) at the moving boundary. Therefore, even in the semilinear case, solutions to (2.18) are not smooth on all of $(0,\sigma) \times (0,\infty)$.

In this section we consider the case when $\phi(u) \equiv du^m$ where $d > 0$ and $m \geqq 2$. Thus, our problem has the form

$$(4.1) \qquad \begin{aligned} & u^j_t(x,t) = d(u^j(x,t))^m_{xx} + f(u^j(x,t)), \quad t > 0, \quad 0 < x < \gamma^j(t), \\ & u^j(x,0) = u_0(x), \quad 0 < x < \sigma, \\ & u^j(\gamma^j(t),t) = (u^j(\gamma^j(t),t))^m_x = 0 \quad \text{if } t > 0, \quad \gamma^j(t) < \sigma, \\ & u^j(x,t) = 0 \quad \text{if } \gamma^j(t) \leqq x \leqq \sigma, \\ & u^j(0,t) = c_0 \quad \text{if } j = D \quad \text{and} \quad (u^j(0,t))^m_x = 0 \quad \text{if } j = N, \end{aligned}$$

where $u_0 \in \mathscr{D}_0$. Recall that if A^j_f is defined by (2.20) and S^j_f is defined by (2.25), then the (generalized) solution u^j, γ^j to (4.1) is defined by

$$(4.2) \qquad \begin{aligned} & u^j(x,t) = [S^j_f(t)u_0](x) \quad \text{for } t \geqq 0, \quad x \in [0,\sigma], \\ & \gamma^j(t) = \inf\{y \in [0,\sigma] : u^j(x,t) = 0 \text{ for } x \in [y,\sigma]\}. \end{aligned}$$

Our first result shows that solutions to (4.1) are $\mathscr{C}^1$ in x.

THEOREM 4. *Suppose that $m \geqq 2$, f is $\mathscr{C}^1$ and u^j is defined by* (4.2). *Then $(u^j(x,t))^m_x$ exists on $(0,\sigma) \times (0,\infty)$ and is continuous in x. In particular, $(u^j(\gamma^j(t),t))^m_x = 0$ for all $t > 0$ such that $\gamma^j(t) < \sigma$.*

For the proof of this theorem we use a preliminary lemma, which follows the methods of Aronson [2] and Kalashnikov [15]. For each $\delta > 0$ the notations $\mathcal{F}_\delta, \mathcal{D}_\delta$, and $S^j_{\delta,f}$ are as in (2.2), (2.3), and (2.15), respectively.

LEMMA 1.3. *Suppose that $\delta > 0$, $v_0 \in \mathcal{D}_\delta$, $f_\delta \in \mathcal{F}_\delta$ is $\mathcal{C}^1$ with $f_\delta(\delta) = 0$, and $v^j(x,t) \equiv [S^j_{\delta,f_\delta}(t)](x)$ where S^j_{δ,f_δ} is defined by (2.15) with $\phi(u) \equiv du^m$ where $m \geqq 2$. Suppose also that $0 < a < b < \sigma$ and $0 < \tau < T$. Then*

$$|(v^j(x,t))^{m-1}_x| \leqq C \qquad \text{for } (x,t) \in (a,b) \times (\tau, T) \tag{4.3}$$

where the constant C depends only on m, a, b and τ. Moreover, if v_0^{m-1} is $\mathcal{C}^2$ on $[0,\sigma]$ then (4.3) remains valid with $\tau = 0$, where C now depends on $\max\{|(v_0(x))^{m-1}_x| : x \in [0,\sigma]\}$ instead of τ.

INDICATION OF PROOF. Since we follow the methods of Aronson [2] and Kalashnikov [15], many of the details of the proof are omitted. For notational convenience set $R = (0,\sigma) \times (0,T)$ and $R^* = (a,b) \times (\tau, T)$. If $v_0 \in \mathcal{C}^\infty \cap D(A^j_{\delta,f_\delta})$ then v^j is a classical solution to

$$\begin{aligned} &v^j_t = (v^j)^m_{xx} + f_\delta(v^j), \quad t > 0, \quad 0 < x < \sigma, \\ &v^j(x,0) = v_0(x), \quad 0 < x < \sigma, \\ &v^j(\sigma,t) = \delta, \quad t > 0, \\ &v^j(0,t) = c_0 \quad \text{if } j = D \quad \text{and} \quad (v^j(0,t))^m_x = 0 \quad \text{if } j = N. \end{aligned} \tag{4.4}$$

Also, $\delta \leqq v^j(x,t) \leqq c_0$ on R by Lemma 4. Define $w = (m/(m-1))v^{m-1}$ so that

$$v = \left(\frac{m-1}{m} w\right)^{1/(m-1)} \equiv \mu(w),$$

and let $g(w) \equiv f_\delta(\mu(w))/\mu(w) = f_\delta(v)/v$. Then

$$w_t = (m-1)ww_{xx} + w_x^2 + (m-1)wg(w).$$

Now for $0 \leqq r \leqq 1$ let $h(r) = Nr(4-r)/3$ where $N = c_0^{m-1}$ and define $p \equiv (h^{-1}(w))_x$. Then, as in [2],

$$\begin{aligned} \frac{1}{2}(p^2)_t - (m-1)hpp_{xx} &= \left[mh'' + (m-1)h\left(\frac{h''}{h'}\right)' \right] p^4 \\ &+ \left[(m+1)h' + 2(m-1)h\frac{h''}{h'} \right] p^2 p_x + (m-1)p\left[\frac{h(w)g(h(w))}{h'(w)}\right]_x. \end{aligned}$$

Now let $\zeta(x,t) \in [0,1]$ be in $\mathcal{C}^2(\bar{R})$ and assume that $\zeta(x,t) \equiv 1$ on R^* and $\zeta(x,t) \equiv 0$ in a neighborhood of the lower and lateral boundaries of R. Also let

$z(x,t)\equiv\zeta(x,t)^2p(x,t)^2$ on R. At a point $(x_0,t_0)\in R$ where z attains its maximum value, we have

$$z_x=2\zeta^2pp_x+2\zeta\zeta_xp^2=0 \quad\text{and}\quad (m-1)hz_{xx}-z_t\leqq 0.$$

This inequality takes the form

$$\begin{aligned}-p^4\zeta^2\left[mh''+(m-1)h\left(\frac{h''}{h'}\right)'\right]&\leqq[\zeta\zeta_t-(m-1)h\zeta\zeta_{xx}+3(m-1)h\zeta_x^2]p^2\\ &\quad-\zeta\zeta_xp^3\left[(m+1)h'+2(m-1)h\frac{h''}{h'}\right]\\ &\quad+(m-1)p\zeta^2\left(\frac{hg(h)}{h'}\right)_x.\end{aligned}\tag{4.5}$$

Since

$$[hg(h)]_h=\frac{1}{m-1}\left[(m-2)\frac{f_\delta}{\mu}+f'_\delta\right]\leqq 0$$

we have that $p(hg(h)/h')_x\leqq 0$, and the last term in (4.5) may be dropped. As in [2], we now obtain that

$$2\zeta^2p^4\leqq C_1p^2+\zeta C_2|p|^3 \quad\text{and}\quad z^2\leqq C_1+C_2^2/4$$

and the assertion of this lemma follows whenever $v_0\in\mathscr{C}^\infty\cap D(A^j_{\delta,f_\delta})$. The general case of $v_0\in\mathscr{D}_\delta$ now follows by $\mathscr{L}^1(0,\sigma)$ approximations.

INDICATION OF THE PROOF OF THEOREM 4. Let $(x_0,t_0)\in(0,\sigma)\times(0,\infty)$ and select a,b,τ and T such that $(x_0,t_0)\in(a,b)\times(\tau,T)$. Let $\delta_n\downarrow 0$ and for each $n\geqq 1$ select a $f_n\in\mathscr{F}_{\delta_n}$ and a $u_n\in D(A^j_{\delta_n,f_n})$ such that $f_n\downarrow f$ uniformly on $(\eta,c_0]$ for each $\eta>0$ and $u_n\downarrow u$ in $\mathscr{L}^1$. It follows from Proposition 1 at the end of §2 that if

$$w^j_n(x,t)\equiv[S^j_{\delta_n,f_n}(t)u_n](x),\qquad t>0,\quad 0\leqq x\leqq\sigma,$$

then $w^j_n(\cdot,t)\downarrow u^j(\cdot,t)$ in $\mathscr{L}^1$ as $n\to\infty$, uniformly for t bounded. By Lemma 13,

$$|(w^j_n(x,t))_x^{m-1}|\leqq C\qquad\text{for all }(x,t)\in(a,b)\times(\tau,T),$$

where C is independent of n (and depends only on a,b,τ and T). As in [2], using the fact that $w^j_n\downarrow u^j$, we can obtain that

$$|(u^j(x,t_0))^{m-1}-(u^j(y,t_0))^{m-1}|\leqq C|x-y|$$

and hence that

$$|u^j(x,t_0)-u^j(y,t_0)|\leqq C_1|x-y|^{(m-1)^{-1}}$$

for all $x, y \in (a, b)$. In particular, if $u^j(y, t_0) = 0$ and $|y - x| \leqq \delta$, then $|u^j(x, t_0)| \leqq C_1 \delta^{(m-1)^{-1}}$. If $u^j(x, t) > 0$ and $t > 0$, it follows as in Oleinik et al. [20] that u^j_x exists and is continuous in x. Thus the same is true for $(u^j)^m_x$. Therefore, it suffices to prove the assertion for points (x_0, t_0) such that $u^j(x_0, t_0) = 0$, and this was done in Aronson [2].

REMARK. Using the fact that the semigroups $S^j_{\delta_n, f_n}$ preserve order in $\mathscr{L}^1$, we can show as in Oleinik et al. [20] that $u^j(x, t) = [S^j_f(t)u_0](x)$ is in fact a classical solution to (4.1) at all points (x, t) such that $u^j(x, t) > 0$.

Our final result shows that the generalized solution to (4.1) is continuous on $(0, \sigma) \times (0, \infty)$ and in fact is Hoelder continuous on each compact subset of $(0, \sigma) \times (0, \infty)$.

THEOREM 5. *Suppose that* $u_0 \in \mathscr{D}_0$, f *is* $\mathscr{C}^1$ *and* $u(x, t) \equiv [S^j_f(t)u_0](x)$ *for* $(x, t) \in [0, \sigma] \times [0, \infty)$. *Then* u *is continuous on* $(0, \sigma) \times (0, \infty)$ *and for each* a, b, τ *and* T *with* $0 < a < b < \tau$ *and* $0 < \tau < T$, *there is a constant* C, *depending only on* a, b, τ *and* T, *such that*

$$
\begin{aligned}
&|u(x, t) - u(y, t)| \leqq C|x - y|^{\beta}, \\
&|u(x, t) - u(x, s)| \leqq C|t - s|^{\beta(\beta+2)^{-1}},
\end{aligned}
\tag{4.6}
$$

for all $x, y \in [a, b]$ *and* $t, s \in [\tau, T]$, *where* $\beta = (m - 1)^{-1}$. *Also,* τ *may be replaced by* 0 *if* $N = \max\{|(u_0(x))^{m-1}_x| : x \in [0, \sigma]\} < \infty$ *and* C *depends on* N.

INDICATION OF PROOF. The first assertion in (4.6) follows from the proof of Theorem 4. To prove the second assertion in (4.6) it suffices to show its validity for appropriate approximations

$$w(x, t) \equiv [S^j_{\delta, f_\delta}(t)u](x), \quad t \geqq 0, \quad 0 \leqq x \leqq \sigma$$

where $\delta > 0$, $\delta \downarrow 0$, $f_\delta \in \mathscr{F}_\delta$, $f_\delta \downarrow f$ and $u_\delta \in D(A^j_{\delta, f_\delta}) \cap \mathscr{C}^\infty$, $u_\delta \downarrow u_0$ in $\mathscr{L}^1$, and the constant C in (4.6) is independent of δ. By Lemma 13 there is a $C_1 > 0$ (independent of δ) such that $|(w(x, t))^{m-1}_x| \leqq C_1$ for $(x, t) \in [a, b] \times [\tau, T]$, and so

$$|w(x, t) - w(y, t)| \leqq \omega(|x - y|)$$

for all $(x, t), (y, t) \in [a, b] \times [\tau, T]$ where $\omega(r) = C^\beta r^\beta$ $[\beta = (m - 1)^{-1}]$. Now we follow the arguments of Kruzhkov [17]. Let $a < x_0 < b$ and $\tau \leqq t_0 \leqq T$ and set $d = \min\{1, x_0 - a, b - x_0\} > 0$. Since $0 < w(x, t) \leqq c_0$ and $|(w)^{m-1}_x| \leqq C_1$, there is a positive constant M such that $mC_1 \leqq M$, $mc_0^{m-1} \leqq M$ and $-f_\delta(c_0) \leqq 4c_0 M$. For $0 < \rho \leqq d$ and $0 < \Delta t < T - t_0$ define

$$Q' = \{(x, t) : t_0 < t < t_0 + \Delta t, |x - x_0| \leqq \rho\}$$

and let Γ denote the lateral and the lower sides of Q'. Also, set $\mu(\rho)=8c_0M/\rho^2$,

$$v^{\pm}(x,t)=w(x_0,t_0)\pm\left[\omega(\rho)+\mu(\rho)(t-t_0)+\frac{2c_0}{\rho^2}(x-x_0)^2\right]$$

and

$$z^{\pm}(x,t)=w(x,t)-v^{\pm}(x,t)$$

for all $(x,t)\in Q'$. It is routine to check that $z^{+}(x,t)\leqq 0$ and $z^{-}(x,t)\geqq 0$ for $(x,t)\in\Gamma$. If

$$L^{+}(v)=m(w)_x^{m-1}v_x+mw^{m-1}v_{xx}-v_t$$

then $L^{+}(w)=-f_\delta(w)\geqq 0$ and $L^{+}(v^{+})\leqq 0$. Thus $L^{+}(z^{+})\geqq 0$ and the maximum principle implies that $z^{+}(x,t)\leqq 0$ in Q'. Thus $w(x,t)\leqq v^{+}(x,t)$ and

$$w(t_0+\Delta t,x_0)-w(t_0,x_0)\leqq\omega(\rho)+\mu(\rho)\Delta t.$$

Define

$$L^{-}(v)=mw^{m-1}v_{xx}-v_t.$$

Then

$$\begin{aligned}L^{-}(z^{-})&=-m(m-1)w^{m-2}(w_x)^2-f_\delta(w)+\frac{4mc_0}{\rho^2}w^{m-1}-\mu(\rho)\\&\leqq -f_\delta(c_0)+M\frac{4c_0}{\rho^2}-\frac{8c_0M}{\rho^2}\\&\leqq -f_\delta(c_0)-4c_0M\\&\leqq 0.\end{aligned}$$

By the maximum principle $z^{-}(x,t)\geqq 0$ in Q' and it follows that

$$w(t_0+\Delta t,x_0)-w(t_0,x_0)\geqq -[\omega(\rho)+\mu(\rho)\Delta t].$$

Therefore,

$$|w(t_0+\Delta t,x_0)-w(t_0,x_0)|\leqq\omega(\rho)+\mu(\rho)\Delta t$$

and so

$$|w(t_0+\Delta t,x_0)-w(t_0,x_0)|\leqq\inf_{0<\rho\leqq d}\{\omega(\rho)+\mu(\rho)\Delta t\}\leqq C_2(\Delta t)^{\beta(\beta+2)^{-1}}$$

and the assertions in Theorem 5 are seen to be true.

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